Mathematics
N-DIMENSIONAL MEDIANS AND CONVEX FUNCTIONS
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Please note that this research is a joint project through the Mathematics and Physics Departments at Purdue University. The classical definition of a median in ∇^1 is defined in the following way: Given a set S, where S={$x_1, x_2, \ldots, x_k$} and $x_1<x_2<\ldots<x_k$, the median is the "middle" term. The idea of a middle term does not work well in more than one dimension for the median will not be preserved through coordinate changes. The median can be described as the point z that minimizes the following function,

$$f(z) = |z-x_1| + |z-x_2| + \ldots + |z-x_k|.$$

Using this definition, the median will be preserved through any kind of coordinate changes, translational, and rotational motion. This definition applies to all spaces ∇^n.

There are several situations where the median can be given in completely general terms, with a few stipulations. However, for any points x_k, where $k\geq 5$, there is no general solution. So the goal of my project was to find an algorithm that would find the median for a given set for k number of points in n dimensions.

I began my research with a model of what happens in ∇^2. With a piece of peg-board, fishing line and equally weighted bolts, I constructed a three dimensional picture of where a median occurs by connecting the strings to a key ring and finding the point on the board where the forces balance. Because physical systems always reduce the amount of energy within the system, the point where the tensional forces on the strings balance is also the median of the system. From this I came up with the equation:

$$F_{net} = |\nabla f(z)|.$$

The potential function for the tensional forces is the definition of the median. I used this idea of forces as the basis for the median algorithm.

The algorithm I used was a steepest-decent method. I modeled the situation just as the peg-board acts: strings of equal tension attached to a central ring (with a radius$\Rightarrow$0). The algorithm keeps simulated time and calculates F_{net} at that time. Assuming dt is not very large, the Momentum Principle tells us that $dp=F_{net}\cdot dt$ (where p is momentum). From this I can calculate the motion of the ring. When the forces balance, the point z is the median. These ideas, although much harder to visualize, do apply to multidimensional cases. The exact same types of calculations are made for vectors in ∇^n, where n>3.

The definition of convexity is the following:

$$f(z_1)(1-t)+f(z_2)(t)\geq f((1-t)z_1+tz_2).$$

It can be shown that each element in the median definition, $|z-x_k|$, is it a convex function. When any convex function is added to another convex function, the result is a convex function. I have also proven that if there are 3 points x_k, then the sum $g(z)=|z-x_1|+|z-x_2|+|z-x_3|$ is a strictly convex function. The fourth Lemma states that any convex function where the limit of f as z approaches infinity is infinity, must have a unique global minimum. This is proven by contradiction. The theorem states that any function of the form $f(z) = |z-x_1| + |z-x_2| + \ldots + |z-x_k|$, must be strictly convex and must have a unique global minimum. This minimum is exactly the point that is given by the algorithm I have created.